A geometric computation of cohomotopy groups in co-degree one

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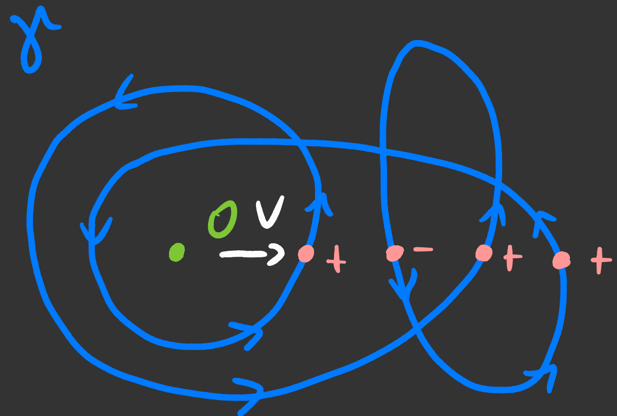
1 Baby example

Q: Curves $\gamma : S^1 \to \mathbb{R}^2 - \{0\}$ up to homotopy?



γ has winding number
$$w(\gamma) = 2.$$

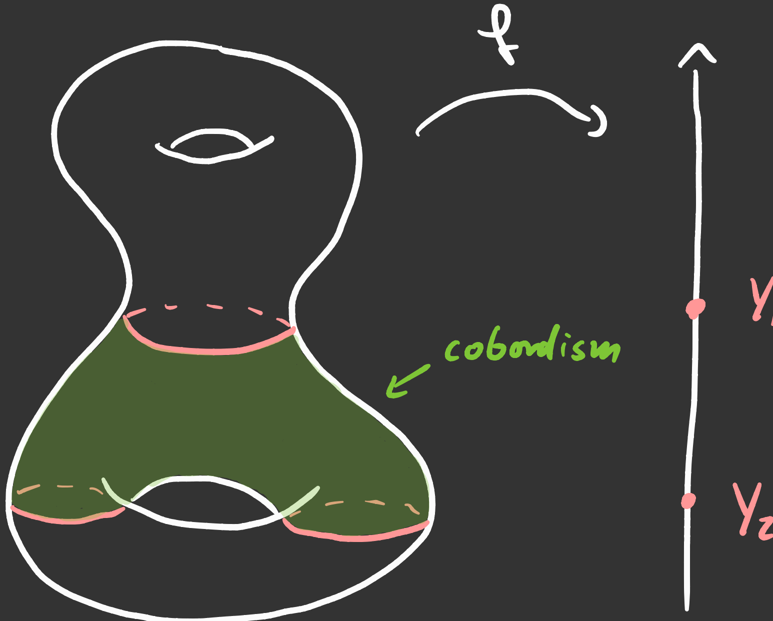
Define $f : S^1 \to S^1$ via
$$f(t) := \frac{\gamma(t)}{\|\gamma(t)\|}.$$

Then: $w(\gamma) = \boxed{\sum_{t \in f^{-1}(v)} \operatorname{sgn} df|_t} = 1 - 1 + 1 + 1 = 2.$

$=: \deg(f; v)$

mapping degree

1 Regular values

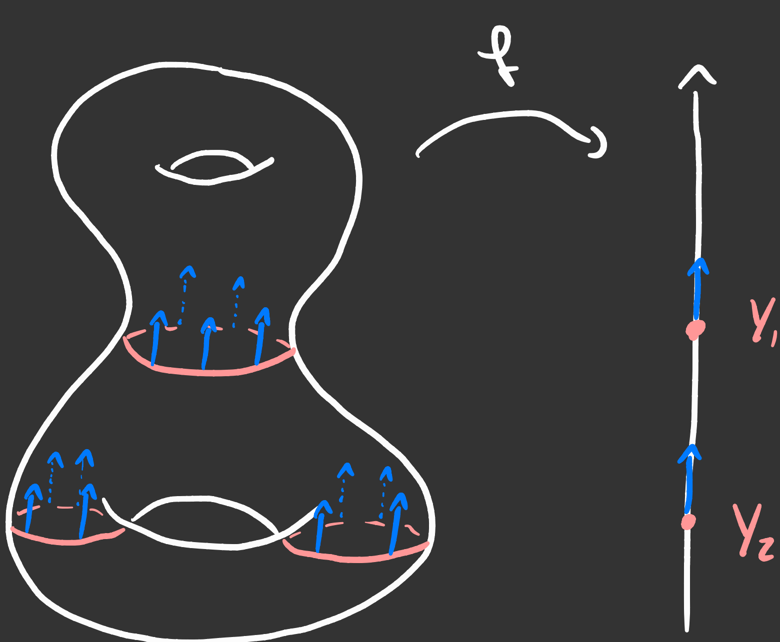


$f: X \longrightarrow M$ smooth

$\Rightarrow f^{-1}(y) \subset X$ submfd.

if y is reg. value

1 Regular values



$f: X \to M$ smooth

$\Rightarrow f^{-1}(y) \subset X$ submfd.

if y is reg. value

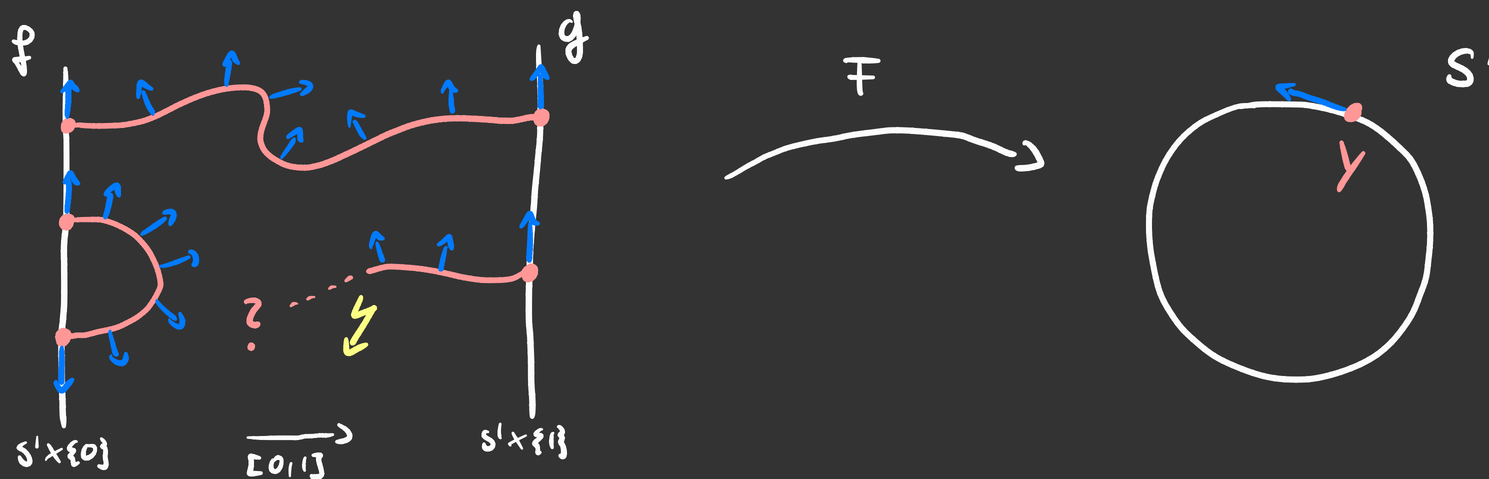
AND $\nu_{f^{-1}(y)} \cong f^*(T_y M)$

$\Rightarrow f^{-1}(y)$ has triv. norm. bundle in X !

Fact: Reg. values are "generic".

Back to the baby example

Q: Is mapping degree a homotopy invariant?

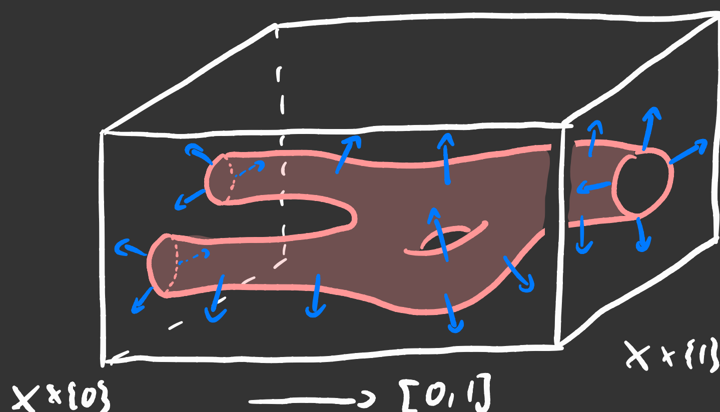


Assume $\deg(f;y) \neq \deg(g;y)$.

1 Pontryagin-Thom construction

Q: Homotopy invariant for $f: X \to S^n$?

Remember: $f^{-1}(y) \subset X$ submfd. $\rightsquigarrow$ cobordisms?



Define:

$$\mathbb{F}_k(X) := \frac{\left\{ \begin{array}{l} k\text{-dim closed } M \subset X \text{ w/} \\ \text{framing of } \nu_M \end{array} \right\}}{\text{normally framed cobord.}}$$

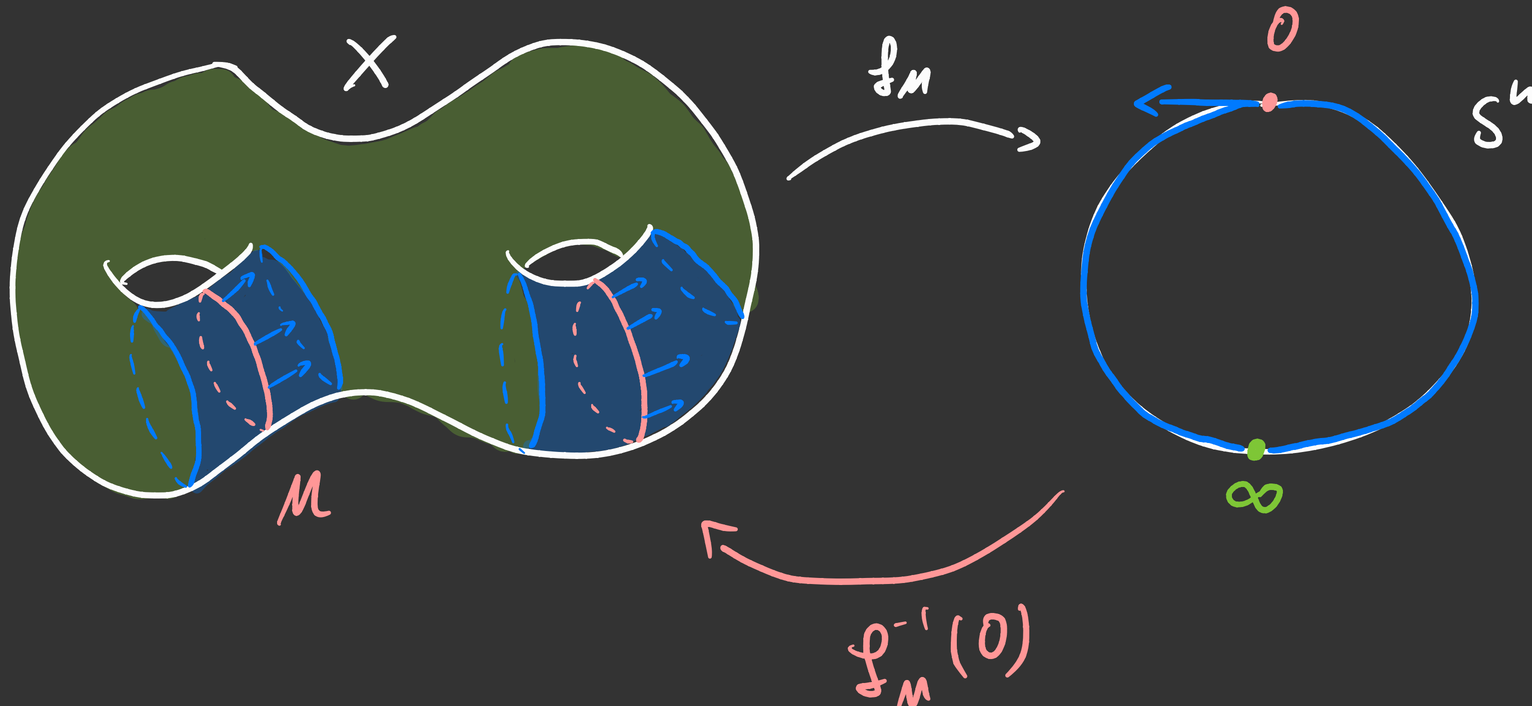
Note: $\mathbb{F}_k(X)$ is group if $\dim X \gg k$:

i) $[M_1, \varphi_1] + [M_2, \varphi_2] = [M_1 \sqcup M_2, \varphi_1 \sqcup \varphi_2]$

ii) $-[M, \varphi] = [M, -\varphi]$ (reverse orientation)

1 Pontryagin-Thom construction

$\underline{A}$: $[X^{n+k}, S^n] \cong \mathbb{F}_k(X)$



what is $[X^{n+1}, S^n]$ (aka $\mathbb{F}_1(X)$)?

- Enumerated by Steenrod '47
- Group ext. by Taylor '09 but purely algebraically
- Geometric proof for $n=3$ and X^4 orientable mfd. by Kirby, Melvin, Teichner '12
- Geometric proof for $n \geq 3$ and X^{n+1} spin mfd. by Konstantis '20

Q: Geometric proof for $n \geq 3$ and X^{n+1} (non-orientable) manifold?

I Twisted homology

$$\mathbb{L}_1(X;\mathbb{Z}w) := \frac{\{\text{Links } L\subset X \text{ w/ orientation of } \nu_L\}}{\text{normally oriented bordisms}}$$

"Algebraically":

$\mathbb{L}_1(X;\mathbb{Z}w) \cong H_1(X;\mathbb{Z}w)$ = first homology w/ "twisted" coefficients

(Thom '54, Atiyah '60)

↳ obvious map $h : \mathbb{F}_1(X) \longrightarrow H_1(X;\mathbb{Z}w)$

forgets framing, remembers orientation

1 The short exact sequence

$$0 \longrightarrow \overset{?}{\ker h} \longrightarrow \mathbb{F}_1(X) \xrightarrow{h} H_1(X; \mathbb{Z}_w) \longrightarrow 0$$

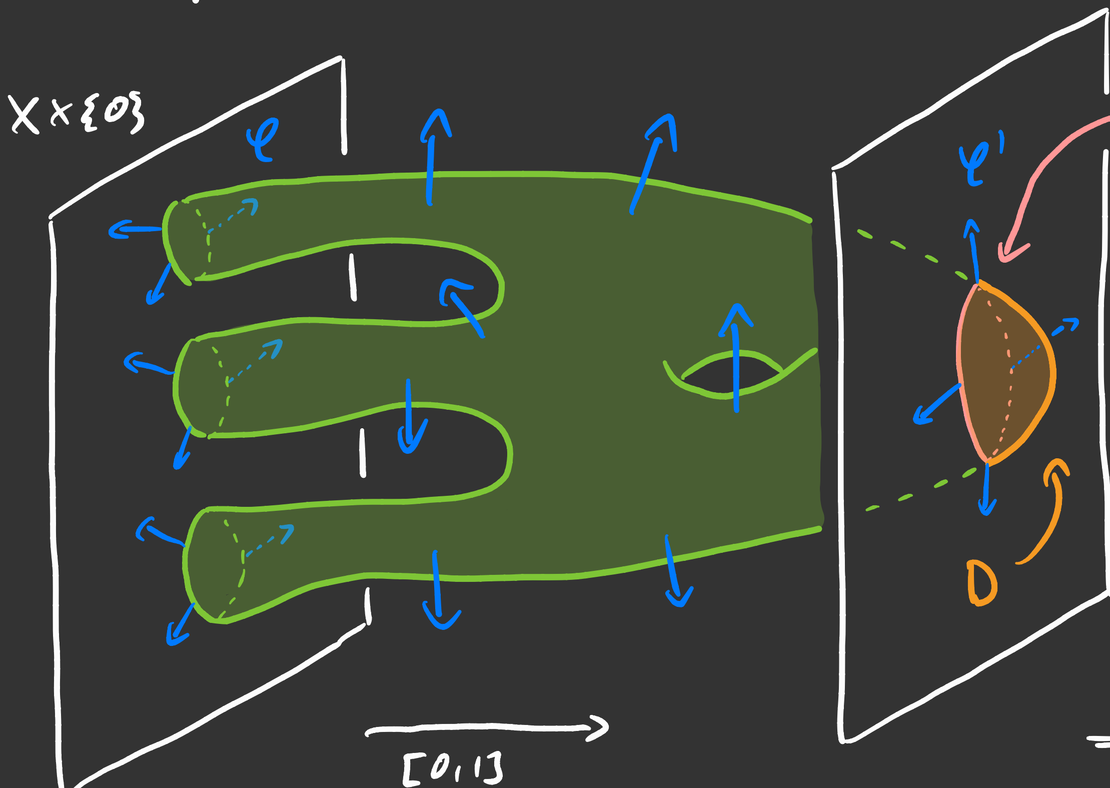
v.b. over 1-dim CW-complex
triv. iff orientable

Also: what is the extension?

| Determine $\operatorname{Ker} h$

Suppose $[L, \varphi] \in \operatorname{Ker} h$.

U contractible



$\rightsquigarrow$ induces (unique) framing φ' over U s.t. $[L, \varphi] = [U, \varphi']$.

__Now:__

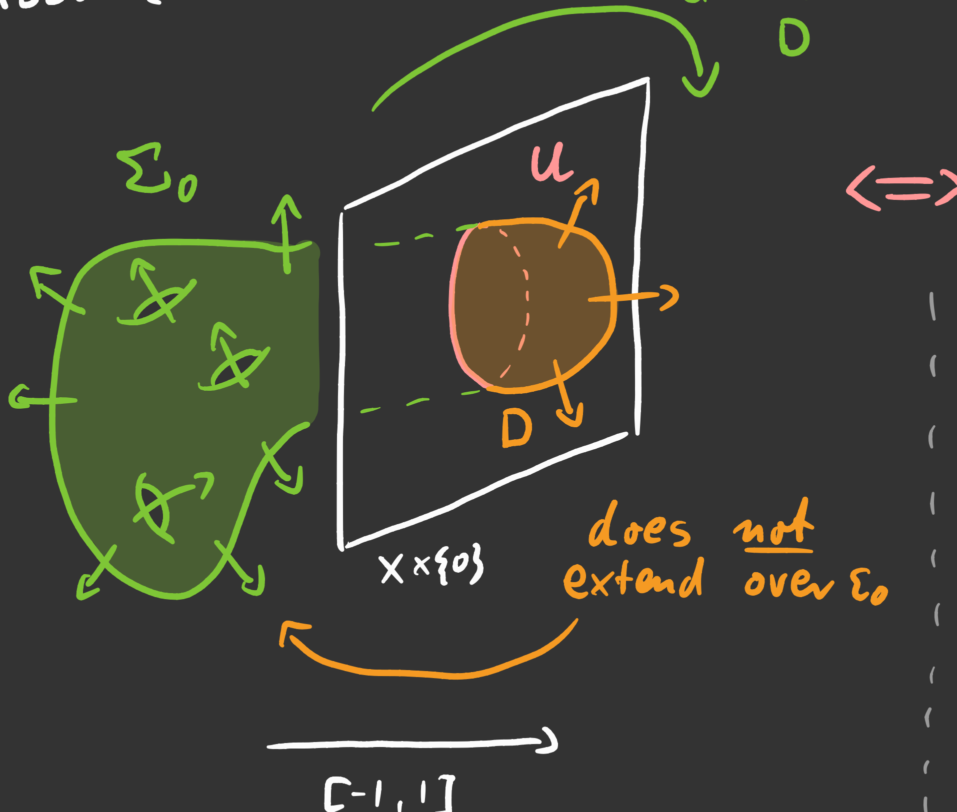
$\pi_1(SO(4)) = \mathbb{Z}_2$

$\Rightarrow$ U has __two__ possible framings (up to homotopy)

$\Rightarrow$ $\operatorname{Ker} h$ is __at most__ $\mathbb{Z}_2$!

Determine $\ker h$

Assume $\ker h = 0$.



$\Sigma = \Sigma_0 \cup D$ surface w/ ν_Σ or. but not triv.

say X is type I iff $\exists\, \Sigma \subset X$ s.t. ν_Σ or. but not triv.

First result

Theorem: X is type I iff $h : \mathbb{F}_1(X) \longrightarrow H_1(X; \mathbb{Z}_w)$ is an isomorphism.

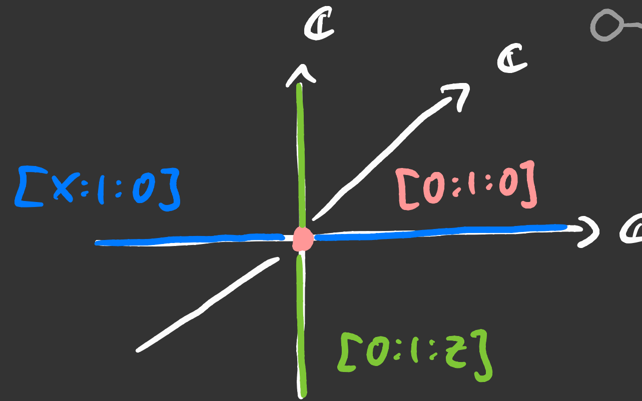
And define:

X is type II iff $\forall \Sigma \subset X$ surface

ν_Σ or. $\Longrightarrow$ ν_Σ triv.

Examples

1) $\mathbb{C}P^1 \subset \mathbb{C}P^2$



$\rightsquigarrow$ $\mathbb{C}P^1$ has odd self-intersection

$\Rightarrow$ $\mathbb{C}P^2$ is of type I

$\rightsquigarrow$ $\mathbb{F}_1(\mathbb{C}P^2) \cong H_1(\mathbb{C}P^2; \mathbb{Z}_w) \cong H_1(\mathbb{C}P^2; \mathbb{Z}) = 0$

2) $\mathbb{R}P^2 \subset \mathbb{R}P^{4k} \xrightarrow{\text{char. classes}}$ $\nu_{\mathbb{R}P^2}$ or. but non-triv.

$\rightsquigarrow$ $\mathbb{F}_1(\mathbb{R}P^{4k}) \cong H_1(\mathbb{R}P^{4k}; \mathbb{Z}_w) \cong 0$

all circles c with orientable ν_c are contractible

Result for type II

Theorem: If X is type II, then $\mathbb{F}_1(X)$ fits into

$$0 \to \mathbb{Z}_2 \to \mathbb{F}_1(X) \to H_1(X;\mathbb{Z}_w) \to 0.$$

The extension is uniquely determined by $\underbrace{w_1^2(X)+w_2(X)}$.

Pin$^-$-obstruction class!

In particular:

$$\mathbb{F}_1(X) \cong H_1(X;\mathbb{Z}_w)\oplus\mathbb{Z}_2 \text{ iff } X \text{ is Pin}^-.$$

Proof idea

Fact: Extension of

$$0 \to \mathbb{Z}_2 \to \mathbb{F}_1(X) \to H_1(X;\mathbb{Z}_w) \to 0$$

is (uniquely) determined by hom.

$$\varepsilon_X : \underbrace{\mathrm{Tor}_2(H_1(X;\mathbb{Z}_w))}_{=: \{[C]\in H_1(X;\mathbb{Z}_w) \mid 2[C]=0\}} \longrightarrow \mathbb{Z}_2 .$$

"measures triviality of ext. in 2-torsion part"

Link to geom. of X:

1) $[C]\in \mathrm{Tor}_2(H_1(X;\mathbb{Z}_w))$ circle.
2) Endow C with normal framing φ.
3) Then $2[C,\varphi]=0$ iff $\varepsilon_X([C])=0$.
4) Build up extension via generators.

Proof idea

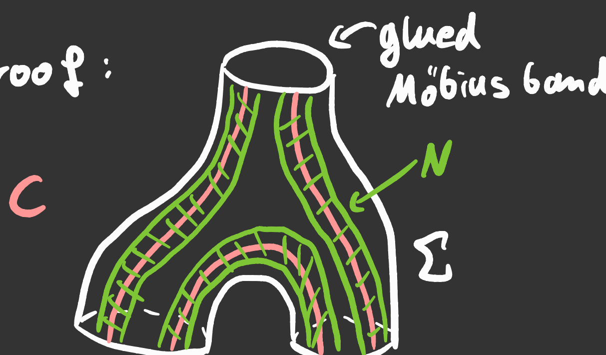
$[C]_{tw} \in \mathrm{Tor}_2(H_1(X;\mathbb{Z}_w))$ generator

$\exists\, \beta : H_2(X;\mathbb{Z}_2) \longrightarrow \mathrm{Tor}_2(H_1(X;\mathbb{Z}_w))$ surjective s.t.

$\beta([\Sigma]_2) = [C]_{tw}$ and C represents $w_1(\nu_\Sigma)$

<u>Prop</u>: $2[C,\varphi] = 0$ iff $w_2(\nu_\Sigma) = 0$

Proof:



$2[C,\varphi] = [\partial N, \psi]$

<u>Now</u>: $[\partial N, \psi] = 0$ iff ψ extends over $\Sigma - N$

<u>Obstruction theory</u>:

ψ extends over $\Sigma - N$ iff $w_2(\nu_\Sigma) = 0$ $\square$

Example

Consider $X = \mathbb{RP}^{n+1}$ and $\mathbb{RP}^2 \subset \mathbb{RP}^{n+1}$

$[\mathbb{RP}^2]$ generates $H_2(X; \mathbb{Z}_2)$ and $\mathrm{Tor}_2(H_1(X; \mathbb{Z}_w)) = H_1(X; \mathbb{Z}_w)$.

$(n+1) \bmod 4$	0	1	2	3
$w_1(\nu_{\mathbb{RP}^2})$	0	1	0	1
$w_2(\nu_{\mathbb{RP}^2})$	1	1	0	0
type	I	II, not Pin^-	II, Pin^-	II, Pin^-
$\pi^n(\mathbb{RP}^{n+1})$	0	$\mathbb{Z}_4$	$\mathbb{Z}_2$	$\mathbb{Z}_2 \oplus \mathbb{Z}_2$

| Pin^- manifolds

X has Pin^--structure, then

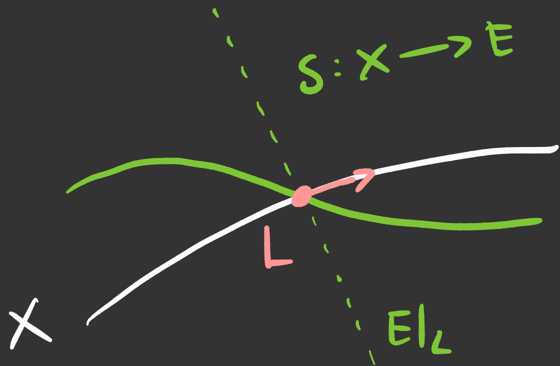
$$\begin{tikzcd}
0 \arrow[r] & \Omega_1^{\mathrm{Pin}^-} \arrow[r] \arrow[d, phantom, "\cong" rotate=-90] & \mathbb{F}_1(X) \arrow[r] \arrow[l, bend left, "\text{induced splitting map}"] & H_1(X;\mathbb{Z}_w) \arrow[r] & 0 \\
& \mathbb{Z}_2 & & &
\end{tikzcd}$$

Then:

$$\begin{tikzcd}
\mathrm{Pin}^-(X) \times H^1(X;\mathbb{Z}_2) \arrow[r, "\mathrm{act}"] \arrow[d] & \mathrm{Pin}^-(X) \arrow[d] \\
\mathrm{split}(X) \times \dfrac{H^1(X;\mathbb{Z}_2)}{\langle w_1(X) \rangle} \arrow[r, "\mathrm{act}"'] & \mathrm{split}(X)
\end{tikzcd}$$

Applications to vector bundles

$\mathbb{R}^n \hookrightarrow E \longrightarrow X^{n+1}$ w/ orientation & spin structure



$L := s^{-1}(0_E)$

spin structure

$\rightsquigarrow \quad \nu_L \cong E|_L \underset{\varphi}{\cong} L \times \mathbb{R}^n$

$\rightsquigarrow \quad [L, \varphi] \in \mathbb{F}_1(X)$

$\mapsto \quad h([L,\varphi]) =$ PD of Euler class $e(E)$

<u>Theorem:</u> E admits a non-vanishing section if and only if $[L, \varphi] = 0$.

<u>Idea:</u> Use null-bordism to "push" s away from zero (similar to Whitney trick).

Applications to vector bundles

Corollary 1: If X is of type I, then E admits a non-vanishing section iff $e(E) = 0$.

Corollary 2: If X is Pin^-, then E admits a non-vanishing section iff $e(E) = 0$ and $\mathfrak{x}([L, e]) = 0$.

splitting map

$\mathfrak{x} : \mathbb{F}_1(X) \longrightarrow \mathbb{Z}_2$

THANK YOU FOR YOUR ATTENTION!