Classification of cohomotopy sets for non-orientable manifolds in co-degree 1

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Motivation

Geometric understanding is essential to many areas of mathematics, and it can lead to new ideas and insights. In this poster, we explore a geometric computation of ***cohomotopy sets*** in co-degree one, a generalization of the mapping degree for maps into spheres. While these sets have been previously understood by Taylor [4] in a purely algebraic manner, our approach offers a novel perspective with geometric constructions. We extend results from Kirby, Melvin, and Teichner [2] to the non-orientable case in dimension four and higher, as well as results from Konstantis [3] to non-orientable Pin^- manifolds.

The Setup

1. X is an $(n+1)$-dimensional closed and connected manifold, $n \geq 3$.
2. $[X, S^n]$ = set of maps $X \to S^n$ up to homotopy.
3. $\mathbb{F}_1(X)$ = cobordism group of embedded links $L \subset X$ with trivialization of ν_L.
4. $H_1(X; \mathbb{Z}_w)$ = first homology of X with local coeff. in the orientation sheaf.

The Pontryagin-Thom construction

There is an isomorphism $\mathbb{F}_1(X) \cong [X, S^n]$ given by the ***Pontryagin-Thom construction***. Here, the map $[X, S^n] \to \mathbb{F}_1(X)$ maps a homotopy class to the preimage of a regular value of S^n. The inverse ***collapse map*** $\mathbb{F}_1(X) \to [X, S^n]$ takes a normally framed submanifold M, and uses the *product neighborhood theorem* to construct a map $f_M\colon X \to S^n$. This construction is illustrated in Figure 1.

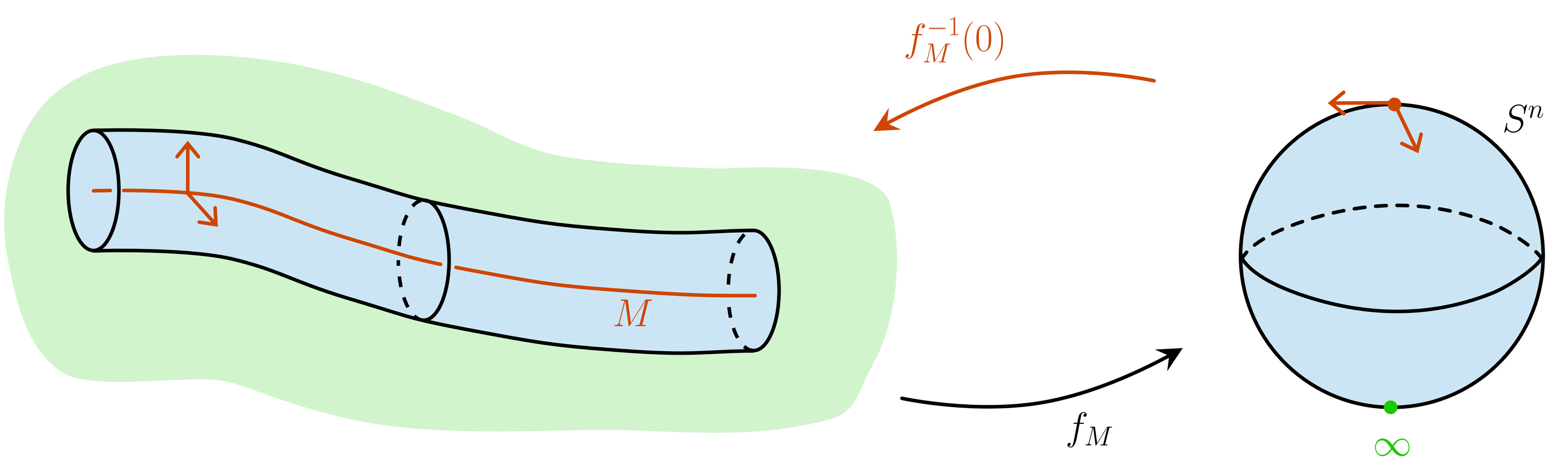

Figure 1. The collapse map.

The Pontryagin-Thom isomorphism is a **refinement of twisted Poincaré duality**:

$$\begin{array}{ccc} [X, S^n] & \xrightarrow{\cong} & \mathbb{F}_1(X) \\ \downarrow & & \downarrow h \\ H^n(X;\mathbb{Z}) & \xrightarrow[\text{twisted PD}]{\cong} & H_1(X;\mathbb{Z}_w). \end{array}$$

Preliminaries

A variation of Thom's seminal work [5] discussed in Atiyah [1] implies that

$$H_1(X;\mathbb{Z}_w) \cong \text{cobordism group of embedded links } L \subset X \text{ with orientation of } \nu_L.$$

Moreover, there is a **forgetful map** $h\colon \mathbb{F}_1(X) \to H_1(X;\mathbb{Z}_w)$ that forgets the framing of ν_L but remembers the orientation.

For the following, we need to distinguish between **two different types of manifolds**:

1. X is type I $\Leftrightarrow \exists$ surface $\Sigma \subset X$ with ν_Σ orientable but non-trivializable.
2. X is type II $\Leftrightarrow \forall$ surfaces $\Sigma \subset X$ with ν_Σ orientable $\Rightarrow \nu_\Sigma$ is trivializable.

The key ingredient to all proofs is that the normal bundle of each orientation-preserving circle has exactly two trivializations up to homotopy because $\pi_1(\mathrm{SO}(n)) = \mathbb{Z}_2$.

Type I Manifolds

A manifold X is of type I if and only if $h\colon \mathbb{F}_1(X) \to H_1(X;\mathbb{Z}_w)$ is an isomorphism.

If $\ker(h)$ is trivial, we can glue both null-cobordisms (one for each trivialization of the circle) to construct a surface Σ that characterizes type I. If we have such a surface Σ, we can cut it into two pieces D and Σ_0 to obtain two null-cobordisms, and thus $\ker(h)$ is trivial. The construction (with suppressed dimensions) is illustrated in Figure 2.

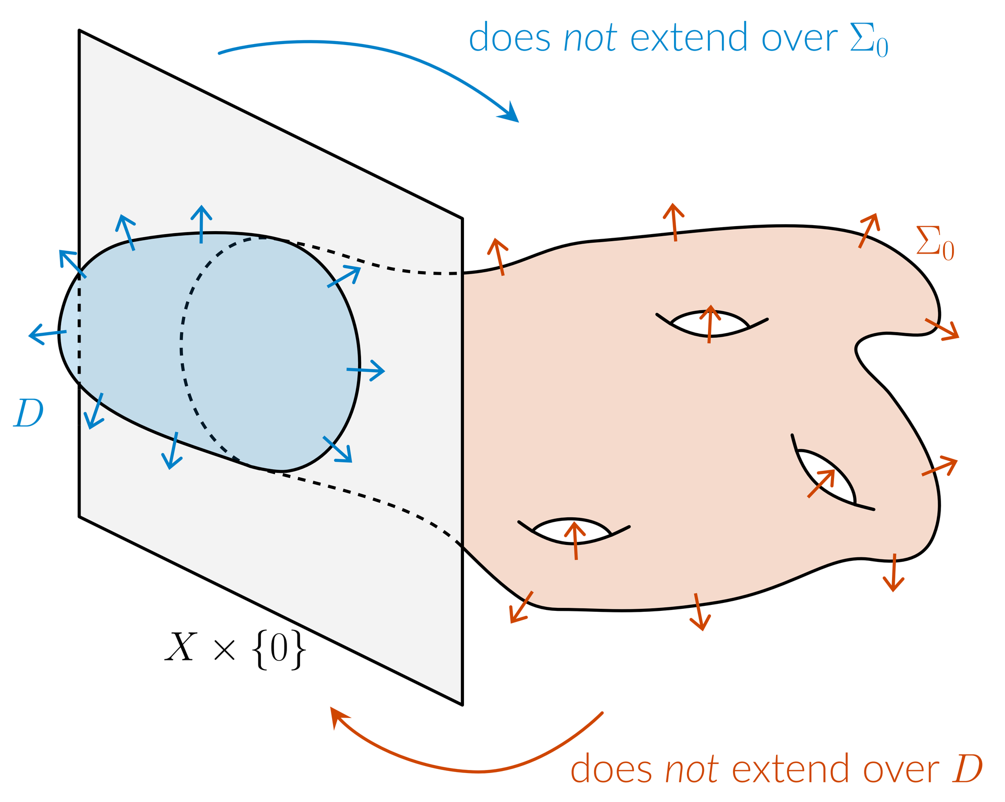

Figure 2. The surface $\Sigma = D \cup \Sigma_0$ with ν_Σ orientable but non-trivializable.

Type II Manifolds

If X is of type II, the framed cobordism group $\mathbb{F}_1(X)$ fits into the short exact sequence

$$0 \longrightarrow \mathbb{Z}_2 \longrightarrow \mathbb{F}_1(X) \xrightarrow{h} H_1(X;\mathbb{Z}_w) \longrightarrow 0. \qquad (1)$$

The associated extension is determined by $w_1^2(X) + w_2(X)$, **the Pin^--obstruction class**. As a special case, we get that this sequence splits if and only if X is Pin^-.

Pin Manifolds

If X is Pin^-, a given Pin^--structure for X determines a splitting map $\mathbb{F}_1(X) \to \mathbb{Z}_2$ of (1). Let us denote

$$\mathcal{S}\mathrm{plit}(X) := \{\text{splitting maps } \mathbb{F}_1(X) \to \mathbb{Z}_2 \text{ of (1)}\},$$
$$\mathcal{P}\mathrm{in}(X) := \{\text{equivalent } \mathrm{Pin}^-\text{-structures of } X\}.$$

We prove that there is a commutative diagram

$$\begin{array}{ccc} \mathcal{P}\mathrm{in}(X) \times H^1(X;\mathbb{Z}_2) & \xrightarrow{\text{act}} & \mathcal{P}\mathrm{in}(X) \\ \downarrow & & \downarrow \\ \mathcal{S}\mathrm{plit}(X) \times H^1(X;\mathbb{Z}_2)/\langle w_1(X)\rangle & \xrightarrow{\text{act}} & \mathcal{S}\mathrm{plit}(X), \end{array}$$

where all group actions are simply transitive. This shows that $\mathcal{P}\mathrm{in}(X) \to \mathcal{S}\mathrm{plit}(X)$ is 2-to-1 if X is non-orientable and $\mathcal{S}\mathrm{pin}(X) \to \mathcal{S}\mathrm{plit}(X)$ is 1-to-1 if X is orientable. Equivalently, a Pin^--structure of X amounts to a choice of trivialization of the normal bundle over each homotopy class of orientation-preserving circles of X – up to $w_1(X)$ action.

Application to Vector Bundles

Suppose X is of type I or Pin^- and $E \to X$ is an spin vector bundle of rank n. The zero locus of a transversal section gives rise to a link $L \subset X$. In particular, $E|_L \cong \nu_L$ and a spin structure on E gives a trivialization φ of ν_L. The element $[L, \varphi] \in \mathbb{F}_1(X)$ defines a **refinement of the Euler class**. That is, $[L, \varphi] = 0$ if and only if E admits a non-vanishing section. The idea is to push the zeros away using the null-cobordism as illustrated in Figure 3.

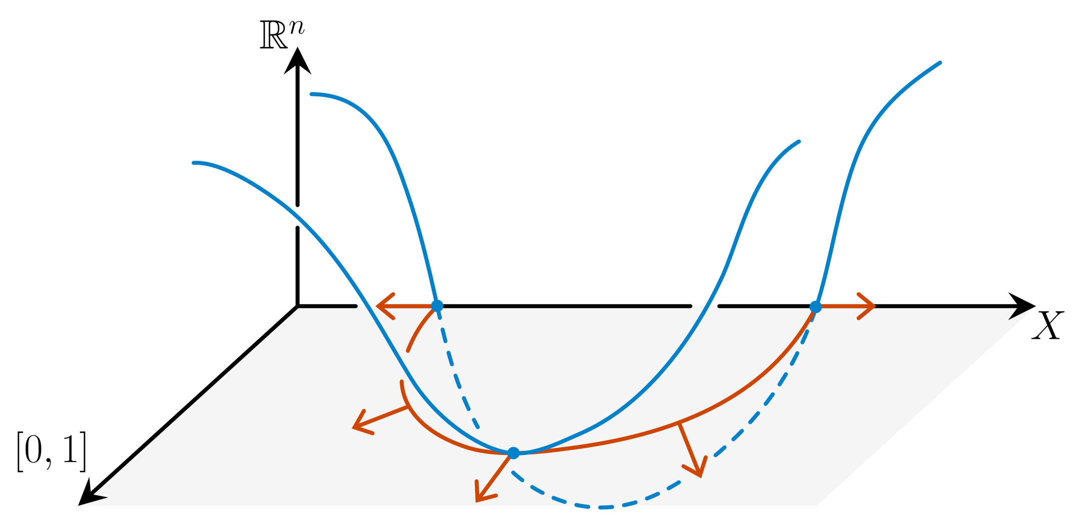

Figure 3. Pushing the zero locus away from zero.

References

[1] M. F. Atiyah. Bordism and cobordism. *Proc. Cambridge Philos. Soc.*, 57:200–208, 1961.

[2] Robion Kirby, Paul Melvin, and Peter Teichner. Cohomotopy sets of 4-manifolds. In *Proceedings of the Freedman Fest*, volume 18 of *Geom. Topol. Monogr.*, pages 161–190. Geom. Topol. Publ., Coventry, 2012.

[3] Panagiotis Konstantis. A counting invariant for maps into spheres and for zero loci of sections of vector bundles. *Abh. Math. Semin. Univ. Hambg.*, 90(2):183–199, 2020.

[4] Laurence R. Taylor. The principal fibration sequence and the second cohomotopy set. In *Proceedings of the Freedman Fest*, volume 18 of *Geom. Topol. Monogr.*, pages 235–251. Geom. Topol. Publ., Coventry, 2012.

[5] René Thom. Quelques propriétés globales des variétés différentiables. *Commentarii Mathematici Helvetici*, 28(1):17–86, 1954.

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