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Some Calculations on Chiral Anomalies using an Index Theorem

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Abstract

We compute the chiral anomaly on two spatially compact globally hyperbolic spacetimes. In particular, we examine a foliation of flat tori within an electrodynamic background and Berger spheres. We apply a formula for the relative chiral charge, derived by Bär and Strohmaier in 2016 using an index formula for the Dirac operator under APS boundary conditions. After reviewing the theorem and some discussions, the calculations are presented.

1 Introduction

In this section, we introduce important definitions and conventions, followed by a review of the theorem by Bär and Strohmaier for the relative chiral charge. Then, we present and shortly discuss the results of the calculations which follow in the next sections. Here, we assume some basic knowledge about Clifford algebras and Dirac operators. However, a brief review of Dirac operators on Riemannian manifolds can be found in the first chapter of [Gin09]. For a full discussion of the general semi-Riemannian case consult [Bau81]. In this article, we restrict ourselves to 4-dimensional connected time-oriented Lorentzian manifolds with signature $(-,+,+,+)$, sometimes referred to as *spacetimes*.

The Setup. Let (X,g) be an oriented globally hyperbolic spatially compact spinable spacetime and $SX \to X$ its spinor bundle. Moreover, let $E \to X$ be a complex vector bundle with Hermitian inner product and metric connection ∇^E, which can describe an electromagnetic field on X. For an oriented orthonormal local frame (e_0, e_1, e_2, e_3), we have the Clifford relation $e_i \cdot e_j + e_j \cdot e_i = -2g(e_i, e_j)$ in the Clifford bundle $\mathbb{C}\mathrm{l}(X)$.

The Chirality operator. Let us introduce the *chirality operator*

$$\gamma_5 := \mathrm{i}\, e_0 \cdot e_1 \cdot e_2 \cdot e_3$$

which is an involutive self-adjoint map $SX \to SX$ via Clifford multiplication. This leads to an eigenspace decomposition

$$SX = S_R X \oplus S_L X \tag{1.1}$$

into a "right-handed" and a "left-handed" part, each of equal dimension. In particular, any $s \in SX$ decomposes into right- and left-handed parts $s_{R/L} \in S_{R/L}X$ being eigenvectors of γ_5 corresponding to the eigenvalues ± 1, i.e.

$$\gamma_5 \cdot s_{R/L} = \pm s_{R/L}.$$

The Dirac operator. The local expression of the classical twisted Dirac operator $\nabla\!\!\!/ : C^\infty(X; SX \otimes E) \to C^\infty(X; SX \otimes E)$ reads

$$\nabla\!\!\!/ = -e_0 \cdot \nabla_\nu^{SX\otimes E} + \sum_{i=1}^{3} e_i \cdot \nabla_{e_i}^{SX\otimes E}.$$

Since X is globally hyperbolic, we can write $X = \mathbb{R} \times \Sigma$ with metric of the form $g = -N^2 \mathrm{d}t^2 + g_t$, where t denotes the standard coordinate of the first component, g_t is a family of Riemanianian metrics on Σ depending on t and N is a nowhere vanishing smooth function on X. In particular, each $\iota_t : \Sigma \hookrightarrow X$ is an embedding onto the smooth spacelike Cauchy hypersurface $\Sigma_t := \{t\} \times \Sigma$ for any $t \in \mathbb{R}$. For the sake of convenience, we make the identifications $(\Sigma_t, g|_{\Sigma_t}) \cong (\Sigma, g_t)$. Now, we choose a global timelike unit vector field ν, slicewise normal to Σ_t, in such a way that (ν, e_1, e_2, e_3) is a local oriented basis of $TX|_{\Sigma_t}$ whenever (e_1, e_2, e_3) is a local oriented frame on Σ. The vector field ν is future-directed and its flow increases the t-variable by the natural choice of time-orientation. Moreover, we identify $S\Sigma_t \cong S_{R/L}X$ with the corresponding Clifford isomorphism $\gamma \mapsto \pm\mathrm{i}\nu \cdot \gamma$. As shown in [Gin09, pp. 19–21], the Dirac operator along Σ_t can then be expressed as

$$\nabla\!\!\!/ = -\nu \cdot \nabla_\nu^{SX\otimes E} - \nu \cdot (\mathrm{i}D_t + 2H)\,. \tag{1.2}$$

Here, H is the mean curvature and D_t reads

$$D_t = \begin{pmatrix} \not{\nabla}_{\Sigma_t} & 0 \\ 0 & -\not{\nabla}_{\Sigma_t} \end{pmatrix},$$

where $\not{\nabla}_{\Sigma_t}$ denotes the classical twisted Dirac operator on Σ_t locally given by

$$\not{\nabla}_{\Sigma_t} = \sum_{i=1}^{3} e_i(t) \cdot \nabla^{S\Sigma_t \otimes E}_{e_i(t)}$$

for an orthonormal oriented local frame $(e_1(t), e_2(t), e_3(t))$ with respect to g_t, where we used the aforementioned identifications. By abuse of notation, we make no distinction between the vector bundle $E \to X$ and its pull-back $\iota_t^* E \to \Sigma_t$, as well as between their connections.

Harmonic Spinors and Weyl equation. A massless physical particle is described by a harmonic spinor $\psi \in \mathrm{Dom}(\not{\nabla}) \subset L^2(X; SX \otimes E)$. In other words ψ satisfies the so called *Dirac equation*

$$\not{\nabla}\psi = 0.$$

In the case of a product structure near Σ_{t_0}, which means that near t_0 the metric is of the form $g = -\mathrm{d}t^2 + g_{t_0}$ and $\nabla^E|_{\Sigma_t} = \iota_{t_0}^*(\nabla^E)$, the mean curvature vanishes near t_0. Identifying spinors by parallel transport along the t-lines, D_t is essentially given by D_{t_0}, and, using eq. (1.2), the Dirac equation reduces with $\nu = \frac{\partial}{\partial t}$ to the so-called *Weyl equation*

$$\mathrm{i}\frac{\partial}{\partial t}\psi = D_{t_0}\psi$$

in a neighborhood of t_0. This equation can locally be solved by spinors of the form

$$\psi = \mathrm{e}^{-\mathrm{i}\lambda t}\psi_\lambda,$$

where ψ_λ is a t-parallel eigenspinor of D_{t_0} corresponding to the eigenvalue λ, i.e.

$$D_{t_0}\psi_\lambda = \lambda\psi_\lambda. \tag{1.3}$$

Regarding the chirality splitting in eq. (1.1), we have

$$-\nu = \begin{pmatrix} 0 & \mathrm{id} \\ \mathrm{id} & 0 \end{pmatrix}$$

via Clifford multiplication, and for any spinor $\varphi \in C^\infty(X; SX \otimes E)$, we find that

$$D_t(\nu \cdot \varphi) = -\nu \cdot D_t(\varphi)$$

along Σ_t. If $\varphi_\lambda \in C^\infty(\Sigma_{t_0}; S_R X \otimes E)$ is a right-handed eigenspinor of D_{t_0} corresponding to the eigenvalue λ then $-\nu\cdot\varphi_\lambda \in C^\infty(\Sigma_{t_0}; S_L X \otimes E)$ is a left-handed eigenspinor corresponding to the eigenvalue $-\lambda$. So physically, λ is the energy at t_0 of a corresponding right-handed particle φ_λ, described just by the following eigenvalue problem on Σ_{t_0}:

$$\nabla\!\!\!\!/_{\Sigma_{t_0}} \varphi_\lambda = \lambda \varphi_\lambda.$$

This is possible due to the identifications we made previously, particularly $S\Sigma_{t_0} \cong S_R X$. Furthermore, note that the spatial compactness of Σ_{t_0} ensures that the spectrum of $\nabla\!\!\!\!/_{\Sigma_{t_0}}$ is discrete and all eigenspaces are of finite dimension and consist of smooth spinors. The solutions $\psi_{\pm\lambda}$ to eq. (1.3) are then given by φ_λ and $-\nu \cdot \varphi_\lambda$ parallelly transported along the t-lines in a neighborhood of the slice Σ_{t_0}.

The Relative Chiral Charge. Let us fix $t_- < t_+$ for which $\Sigma_\pm := \Sigma_{t_\pm}$ are two Cauchy hypersurfaces with product structure in their neighborhoods and consider its corresponding vacuum states $\omega_{\Sigma_\pm}$. What we are interested in is the *relative chiral charge*, denoted by $Q_{\mathrm{chir}}^{\omega_{\Sigma_-}, \omega_{\Sigma_+}}$. This is defined as the sum of the spectral flows of the positive left-handed and negative right-handed Dirac operator between these two hypersurfaces $\Sigma_\pm$, namely

$$Q_{\mathrm{chir}}^{\omega_{\Sigma_-}, \omega_{\Sigma_+}} = 2\,\mathrm{sf}\left(-\nabla\!\!\!\!/_{\Sigma_t}\right)_{t\in[t_-,t_+]}.$$

The spectral flow $\mathrm{sf}\left(-\nabla\!\!\!\!/_{\Sigma_t}\right)_{t\in[t_-,t_+]}$ describes, informally speaking, the net number of eigenvalues of $-\nabla\!\!\!\!/_{\Sigma_t}$ passing the zero line and is closely related to the index of $\nabla\!\!\!\!/$. A rigorous definition of the spectral flow can be found in [APS76] or [Phi96]. Strictly speaking, in the general case, the Dirac operator $\nabla\!\!\!\!/$ on X is not Fredholm and has no well-defined index. But under certain circumstances, such as when imposing APS boundary conditions, $\nabla\!\!\!\!/$ is Fredholm and its index is given by an adapted Atiyah–Patodi–Singer index formula, cf. [BS15]. The formula for the relative chiral charge can be reduced to such a situation and is specified in the following theorem from [BS16].

Theorem 1.1 (Formula for the relative chiral charge)**.** *Let (X, g) be an even-dimensional globally hyperbolic Lorentzian spin manifold, spatially compact, and $\Sigma_-, \Sigma_+ \subset X$ be two spacelike smooth Cauchy hypersurfaces for which Σ_- lies in the past of Σ_+. Furthermore, let $E \to X$ be a Hermitian vector bundle with compatible connection ∇^E. Now, consider the submanifold $M := J^+(\Sigma_-) \cap J^-(\Sigma_+)$ and assume g as well as ∇^E having product structure near its boundary $\Sigma_- \sqcup \Sigma_+$. Then, the relative chiral charge of the vacuum states ω_{Σ_-} and ω_{Σ_+} is given by*

$$\begin{aligned} Q_{chir}^{\omega_{\Sigma_-},\omega_{\Sigma_+}} = 2 \int_M \widehat{A}\left(\nabla^{TX}\right) \wedge \mathrm{ch}\left(\nabla^E\right) \\ - h(\not{\nabla}_{\Sigma_-}) + h(\not{\nabla}_{\Sigma_+}) - \eta(\not{\nabla}_{\Sigma_+}) + \eta(\not{\nabla}_{\Sigma_-}), \end{aligned} \tag{1.4}$$

where $\not{\nabla}_{\Sigma_\pm}$ denotes the classical Dirac operator on $\Sigma_\pm$, $h(\not{\nabla}_{\Sigma_\pm})$ its kernel dimension and $\eta(\not{\nabla}_{\Sigma_\pm})$ the corresponding eta invariant.[1]

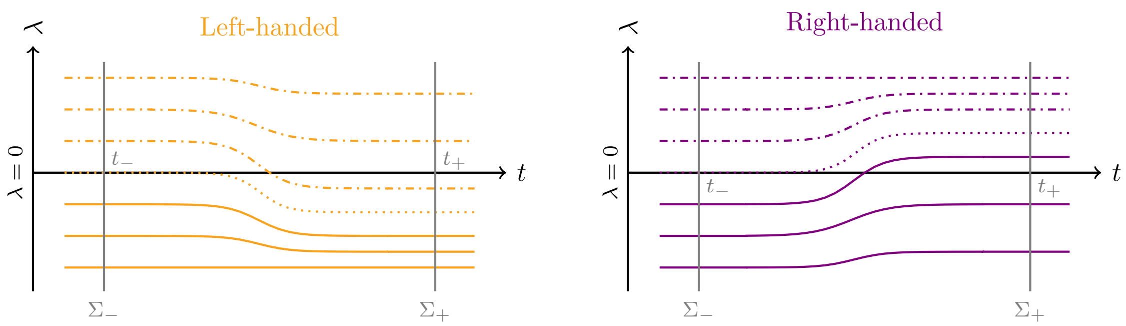

Figure 1: This is a sketched illustrative example of an occurring anomaly. At the slices $t_\pm$, we can see the states with energy eigenvalues λ. Both are within a product structured neighborhood. In between there is some stuff going on resulting in a relative chiral charge of $Q_{\mathrm{chir}}^{\omega_{\Sigma_-},\omega_{\Sigma_+}} = -4$. Notice that zero eigenvalues are treated as negative in the right-handed and positive in the left-handed case.

Remember, the eta function η_Σ of the classical Dirac operator $\not{\nabla}_\Sigma$ on a compact Riemannian manifold Σ is defined as

$$\eta_\Sigma(s) = \sum_{\lambda \in \mathrm{Spec}(\not{\nabla}_\Sigma)\setminus\{0\}} \mathrm{sign}(\lambda)\, |\lambda|^{-s}, \quad \mathrm{Re}(s) \gg 0. \tag{1.5}$$

[1]Note that our sign conventions for $Q_{\mathrm{chir}}^{\omega_{\Sigma_-},\omega_{\Sigma_+}}$ and $\not{\nabla}_{\Sigma_t}$ differ from the one used by Bär and Strohmaier.

It can be shown that η_Σ is regular at $s = 0$ even for any elliptic differential operator, and not only the Dirac operator. One then defines the *eta invariant* as $\eta(\not{\nabla}_\Sigma) := \eta_\Sigma(0)$. Roughly speaking, the eta invariant measures how "symmetric" the spectrum actually is.

Now, a *chiral anomaly* occurs if the relative chiral charge is not conserved, i.e. $Q_{\text{chir}}^{\omega_{\Sigma_-},\omega_{\Sigma_+}} \neq 0$. An illustrative example of this case is sketched in Figure 1.

This Article. In this article we consider two curved manifolds in which chiral anomalies can take place. More specifically, we investigate spatial foliations of flat tori within an electromagnetic background and 3-spheres with Berger metric. They are quintessentially of great interest as they represent two non-trivial examples of causing a chiral anomaly. The study of these spaces reveals that the torus admits a symmetric spectrum while the Berger sphere consists of an asymmetric spectrum. A consequence will be that the chiral anomaly in the former case is caused only by an external electric field, whereas, in the latter case, gravity itself is perpetrator of the anomaly. The corresponding calculations are done in the following sections. The final results can be found in eq. (2.10) as well as eq. (3.22) and are summarized below once again:

Manifold M	Metric	Spectrum	External field	$Q_{\text{chir}}^{\omega_{\Sigma_-},\omega_{\Sigma_+}}$
$[t_-, t_+] \times \mathbb{T}^3$	spatially flat	symmetric	yes	2
$[t_-, t_+] \times S^3$	Berger	asymmetric	no	4

The calculations in this article of the Dirac spectrum on $\mathbb{T}^3$ are based on [Gin09, pp. 29–31], slightly adapted to account for an external field. The Dirac spectrum on Berger's sphere was originally calculated by [Hit74] and later generalized by [Bär96] for higher dimensions. However, in this article, we will restrain ourselves to the calculations in [Hit74]. Using analytic number theory, [Hab00] computed the eta invariant for general Berger spheres. We will proceed here in a similar way. An alternative and purely geometrical approach in the 3-dimensional case came from [Koh04]. He identified Berger spheres as spheres sitting in the complex projective space $\mathbb{CP}^2$ and derived the eta invariant from the Atiyah–Singer–Patodi index theorem.

2 Flat Tori with External Field

Let $\{b_1, b_2, b_3\}$ be any basis of $\mathbb{R}^3$ and $\{e_1, e_2, e_3\}$ its standard basis. Furthermore, let $\Gamma = \left\{\sum_{i=1}^3 c_i\, b_i \mid c_i \in \mathbb{Z}\right\}$ be the corresponding lattice. We define the resulting torus as $\mathbb{T}^3 := \mathbb{R}^3/\Gamma$.

Now, we consider $X := \mathbb{R} \times \mathbb{T}^3$ with metric

$$g := -\mathrm{d}t^2 + a_1^2(t)\left(\mathrm{d}x^1\right)^2 + a_2^2(t)\left(\mathrm{d}x^2\right)^2 + a_3^2(t)\left(\mathrm{d}x^3\right)^2,$$

where $a_i : \mathbb{R} \to \mathbb{R}$ are positive smooth functions and (t, x^1, x^2, x^3) denote the standard coordinates of $\mathbb{R} \times \mathbb{R}^3$. We put $\Sigma_t := \{t\} \times \mathbb{T}^3$, which is a compact Cauchy hypersurfaces with Riemannian metric

$$g|_{\Sigma_t} = a_1^2(t)\left(\mathrm{d}x^1\right)^2 + a_2^2(t)\left(\mathrm{d}x^2\right)^2 + a_3^2(t)\left(\mathrm{d}x^3\right)^2$$

for each $t \in \mathbb{R}$. We consider now $t_- < t_+$ and define $\Sigma_\pm := \Sigma_{t_\pm}$ so that $M = [t_-, t_+] \times \mathbb{T}^3$. Here, let E be the trivial complex line bundle with connection $\nabla^E := \partial + \mathrm{i}A$ and electromagnetic potential A of the form $A = A_1(t)\mathrm{d}x^1 + A_2(t)\mathrm{d}x^2 + A_3(t)\mathrm{d}x^3$. Therefore, E describes an external electric field with no magnetic components. Furthermore, we assume the functions a_j and A_j $(j = 1, 2, 3)$ to be locally constant near $t_\pm$ such that everything has product structure near $\Sigma_- \sqcup \Sigma_+$.

The Integral Term. In order to facilitate computations, we first examine the $\widehat{A}(\nabla^{TX}) \wedge \mathrm{ch}(\nabla^E)$-integrand. An oriented orthonormal global frame with respect to g is given by $\xi_0 := \frac{\partial}{\partial t}$ and $\xi_i := \frac{1}{a_i}\frac{\partial}{\partial x^i}$ with $i = 1, 2, 3$. Computation of its Lie brackets yields

$$[\xi_i, \xi_j] = 0, \tag{2.1}$$

$$[\xi_i, \xi_0] = \frac{\dot{a}_i}{a_i}\,\xi_i \tag{2.2}$$

for $i, j = 1, 2, 3$. Since we are dealing with an orthonormal frame, the connection 1-form ω can be calculated in the following way:

$$\begin{aligned} 2\omega^i_j(\xi_k) &= 2g\left(\nabla^{TX}_{\xi_k}\xi_j, \xi_i\right) \\ &= g\left(\xi_i, [\xi_k, \xi_j]\right) - g\left(\xi_j, [\xi_k, \xi_i]\right) - g\left(\xi_k, [\xi_j, \xi_i]\right), \end{aligned} \tag{2.3}$$

$i, j, k \in \{0, 1, 2, 3\}$. To obtain this, we applied the Koszul formula, in which the derivation terms vanish as we evaluate orthonormal vector fields. Due to eqs. (2.1) & (2.2) we have

$$\omega = \begin{pmatrix} 0 & -\dot{a}_1\,\mathrm{d}x^1 & -\dot{a}_2\,\mathrm{d}x^2 & -\dot{a}_3\,\mathrm{d}x^3 \\ \dot{a}_1\,\mathrm{d}x^1 & 0 & 0 & 0 \\ \dot{a}_2\,\mathrm{d}x^2 & 0 & 0 & 0 \\ \dot{a}_3\,\mathrm{d}x^3 & 0 & 0 & 0 \end{pmatrix}.$$

The curvature 2-form Ω can be calculated using $\Omega = \mathrm{d}\omega + \omega \wedge \omega$, and therefore,

$$\Omega = \begin{pmatrix} 0 & \ddot{a}_1\,\mathrm{d}x^1 \wedge \mathrm{d}t & \ddot{a}_2\,\mathrm{d}x^2 \wedge \mathrm{d}t & \ddot{a}_3\,\mathrm{d}x^3 \wedge \mathrm{d}t \\ \ddot{a}_1\,\mathrm{d}t \wedge \mathrm{d}x^1 & 0 & \dot{a}_2\dot{a}_1\,\mathrm{d}x^2 \wedge \mathrm{d}x^1 & \dot{a}_2\dot{a}_1\,\mathrm{d}x^3 \wedge \mathrm{d}x^1 \\ \ddot{a}_2\,\mathrm{d}t \wedge \mathrm{d}x^2 & \dot{a}_1\dot{a}_2\,\mathrm{d}x^1 \wedge \mathrm{d}x^2 & 0 & \dot{a}_3\dot{a}_2\,\mathrm{d}x^3 \wedge \mathrm{d}x^2 \\ \ddot{a}_3\,\mathrm{d}t \wedge \mathrm{d}x^3 & \dot{a}_1\dot{a}_3\,\mathrm{d}x^1 \wedge \mathrm{d}x^3 & \dot{a}_2\dot{a}_3\,\mathrm{d}x^2 \wedge \mathrm{d}x^3 & 0 \end{pmatrix}. \tag{2.4}$$

Moreover, the Pontryagin class $p(\nabla^{TX})$ is given by

$$p\left(\nabla^{TX}\right) = \det\left(1 + \frac{1}{2\pi}\Omega\right). \tag{2.5}$$

Using the Leibniz formula, we obtain that

$$\det(A) = \sum_{\sigma \in S_n} \operatorname{sign}(\sigma) \prod_{i=1}^{n} a_{\sigma(i),i}$$

for any $(n \times n)$-matrix A. On comparing with eq. (2.4), we can see that the only term surviving in eq. (2.5) is the constant 1 since we deal with wedge products. Hence we determine $p(\nabla^{TX}) = 1$ and thus $\widehat{A}(\nabla^{TX}) = 1$. Because the curvature 2-form of E is simply given by a (1×1)-matrix $F = \mathrm{d}A$, the highest non-zero part of the Chern character $\mathrm{ch}(\nabla^E)$ is a 2-form, and so the volume part of $\widehat{A}(\nabla^{TX}) \wedge \mathrm{ch}(\nabla^E)$ vanishes identically so that

$$\int_M \widehat{A}\left(\nabla^{TX}\right) \wedge \mathrm{ch}\left(\nabla^E\right) = 0. \tag{2.6}$$

The Dirac Spectrum. Since $\mathbb{T}^3$ is parallelizable, X is parallelizable and admits at least one spin structure. The space $\mathbb{T}^3$ comes with exactly 8 different spin structures induced by the lattice Γ acting on $\mathbb{R}^3$ as orientation-preserving isometries. Namely, let $\varepsilon \in \{0,1\}^3$, $\delta_\varepsilon := \sum_{k=1}^3 \varepsilon_k\, b_k \in \Gamma$ and $\langle \cdot, \cdot \rangle_\Gamma$ be the induced scalar product for which the $\{b_1, b_2, b_3\}$ are orthonormal, then $\mathrm{Spin}_\varepsilon(T\mathbb{T}^3) = \left(\mathbb{R}^3 \times \mathrm{Spin}(3)\right) / \sim_\varepsilon$ with $(x, a) \sim_\varepsilon (x', a')$ iff $x = x' + \gamma$ and $a = (-1)^{\langle \gamma, \delta_\varepsilon \rangle_\Gamma}\, a'$ for some $\gamma \in \Gamma$. Thus, we can identify spinor fields on $\mathbb{T}^3$ by considering $\varphi \in \Gamma(S\mathbb{R}^3)$ satisfying the periodicity condition

$$\varphi(x+\gamma) = (-1)^{\langle \gamma, \delta_\varepsilon \rangle_\Gamma}\, \varphi(x) \tag{2.7}$$

for all $x \in \mathbb{R}^3$ and any $\gamma \in \Gamma$.

Consider an orthonormal basis $(s_\alpha)_{\alpha=1,2}$ of $S_0\mathbb{R}^3 \cong \mathbb{C}^2$, trivially extended onto $S\mathbb{R}^3$. For any $\gamma \in \Gamma$ we define

$$\varphi_{\alpha,\gamma}(x) := \mathrm{e}^{2\pi \mathrm{i}\left(\langle x,\gamma\rangle_\Gamma + \frac{1}{2}\langle x,\delta_\varepsilon\rangle_\Gamma\right)}\, s_\alpha \tag{2.8}$$

and it is easy to see that (2.7) is indeed satisfied. A global orthonormal frame with respect to $g|_{\Sigma_t}$ is given by $E_k(t) := \frac{1}{a_k(t)}\frac{\partial}{\partial x^k}$ with $k = 1, 2, 3$. Since the Levi-Civita connection on Σ_t is trivial, the covariant derivative on $S\Sigma_t \otimes E$ becomes

$$\nabla^{S\Sigma_t\otimes E}_{E_k(t)}\, \varphi_{\alpha,\gamma} = \frac{2\pi \mathrm{i}}{a_k(t)} \left(\langle e_k, \gamma\rangle_\Gamma + \frac{1}{2}\langle e_k, \delta_\varepsilon\rangle_\Gamma + A_k(t)\right) \varphi_{\alpha,\gamma}.$$

For the twisted Dirac operator, we obtain

$$\begin{aligned} \not{\nabla}_{\Sigma_t}\, \varphi_{\alpha,\gamma} &= \sum_{k=1}^{3} E_k(t)\cdot \nabla^{S\Sigma_t\otimes E}_{E_k(t)}\, \varphi_{\alpha,\gamma} \\ &= 2\pi \mathrm{i} \underbrace{\sum_{k=1}^{3} \frac{1}{a_k(t)} \left(\langle e_k, \gamma\rangle_\Gamma + \frac{1}{2}\langle e_k, \delta_\varepsilon\rangle_\Gamma + A_k(t)\right) E_k(t)}_{=:\,\Lambda_\gamma(t)} \cdot \varphi_{\alpha,\gamma}. \end{aligned} \tag{2.9}$$

To perform the computation, we need to consider two cases. First, take $\Lambda_\gamma(t) = 0$, which is possible only if $A_k(t) = \langle e_k, \gamma\rangle_\Gamma + \frac{1}{2}\langle e_k, \delta_\varepsilon\rangle_\Gamma$ for all $k = 1, 2, 3$. Since $\{e_1, e_2, e_3\}$ is linearly independent, there is at most one $\gamma \in \Gamma$ satisfying this. In that case we deduce 0 is an eigenvalue of at least multiplicity 2. Second, assume $\Lambda_\gamma(t) \neq 0$. Then,

$$\frac{\mathrm{i}\,\Lambda_\gamma(t)}{\|\Lambda_\gamma(t)\|_{g|_{\Sigma_t}}} =: \widehat{\Lambda}_\gamma(t) \;:\; S_0\mathbb{R}^3 \longrightarrow S_0\mathbb{R}^3$$

defines a linear, involutive and unitary map via Clifford multiplication and by diagonalisation, we obtain an orthogonal decomposition

$$S_0\mathbb{R}^3 = \mathrm{Ker}\left(\widehat{\Lambda}_\gamma(t) - \mathrm{id}\right) \oplus \mathrm{Ker}\left(\widehat{\Lambda}_\gamma(t) + \mathrm{id}\right).$$

In fact, both kernels have the same dimension. This can be seen as follows. Choose a unit vector $v \in T_0\mathbb{R}^3$ orthogonal to $\Lambda_\gamma(t)$, relative to the metric $g|_{\Sigma_t}$. Then $s \mapsto v \cdot s$ gives an isomorphism between these kernels and so each of them has dimension 1. Now, regarding the orthonormal basis $(s_\alpha)_{\alpha=1,2}$ in eq. (2.8), we choose unit vectors $\sigma_\gamma^\pm(t) \in \operatorname{Ker}\Big(\widehat{\Lambda}_\gamma(t) \mp \mathrm{id}\Big)$ and finally get

$$\not{\nabla}_{\Sigma_t}\, \varphi_\gamma^\pm(t) = \pm 2\pi \left\|\Lambda_\gamma(t)\right\|_{g|_{\Sigma_t}} \varphi_\gamma^\pm(t)$$

from eq. (2.9). Therefore $\lambda_\gamma^\pm(t) = \pm 2\pi \left\|\Lambda_\gamma(t)\right\|_{g|_{\Sigma_t}}$ are non-zero eigenvalues of multiplicity at least 1.

Knowing that $\{x \mapsto \mathrm{e}^{2\pi i \langle x,\gamma\rangle_\Gamma} \mid \gamma \in \Gamma\}$ is a Hilbert basis of $L^2(\mathbb{T}^3, \mathbb{C})$, we conclude that $\Big(\varphi_\gamma^\pm(t)\Big)_{\gamma\in\Gamma}$ is a Hilbert basis of $L^2(\Sigma_t; S\Sigma_t \otimes E)$ and we have indeed obtained all eigenvalues with no higher multiplicities as noted. We summarize:

Theorem 2.1 (Dirac Spectrum of the 3-Torus)**.** *Assume the conditions stated at the beginning of this section. Then, the eigenvalues of the classical twisted Dirac operator* $\not{\nabla}_{\Sigma_t}$ *on* Σ_t *are given by*

$$\lambda_\gamma^\pm(t) = \pm 2\pi \sqrt{\sum_{k=1}^{3} \frac{1}{a_k^2(t)} \left(\langle e_k, \gamma\rangle_\Gamma + \frac{1}{2}\langle e_k, \delta_\varepsilon\rangle_\Gamma + A_k(t)\right)^2}$$

for any $\gamma \in \Gamma$*. Each non-zero eigenvalue comes with multiplicity* 1*. In the case* $\lambda_\gamma(t) = 0$*, its multiplicity is* 2*.*

Eta Invariant and Result. Finally, we are interested in the eta function η_{Σ_t}, whose formula is given by eq. (1.5). Due to the eigenvalue symmetry about zero (cf. Theorem 2.1), we obtain $\eta_{\Sigma_t}(s) = 0$ for $\operatorname{Re}(s) \gg 0$. Meromorphic continuation yields $\eta(\not{\nabla}_{\Sigma_t}) = 0$, independent of $t \in \mathbb{R}$.

Neither the integral term in eq. (2.6) nor the eta invariant contribute to the relative chiral charge. Consequently, an anomaly can occur only if the electric field alters the dimension of the kernels $h(\not{\nabla}_{\Sigma_\pm})$. For each $k = 1, 2, 3$, we put $A_k(t_+) := -\frac{1}{2}\langle e_k, \delta_\varepsilon\rangle_\Gamma$ so that $h(\not{\nabla}_{\Sigma_+}) = 2$ and $A_k(t_-) := \frac{1}{2}\langle e_k, \delta_{\bar\varepsilon}\rangle_\Gamma$ so that $h(\not{\nabla}_{\Sigma_-}) = 0$. Here $\bar\varepsilon$ denotes the "complement" of ε, i.e. interchanging ones and zeros. Then, using formula (1.4), we finally have

$$Q_{\mathrm{chir}}^{\omega_{\Sigma_-}, \omega_{\Sigma_+}} = 2. \tag{2.10}$$

3 Berger Spheres

The 3-sphere $S^3 \subset \mathbb{C}^2$ can be seen as the Lie group $\mathrm{SU}(2)$ via

$$(z,w) \mapsto \begin{pmatrix} z & w \\ -\bar{w} & \bar{z} \end{pmatrix}.$$

Its Lie algebra $\mathfrak{su}(2)$ consists of (2×2)-skew-Hermitian matrices spanned by

$$X_1 = \begin{pmatrix} \mathrm{i} & 0 \\ 0 & -\mathrm{i} \end{pmatrix}, \quad X_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad X_3 = \begin{pmatrix} 0 & \mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \tag{3.1}$$

obviously obeying the commutator relations

$$[X_3, X_1] = 2X_2, \quad [X_1, X_2] = 2X_3, \quad [X_2, X_3] = 2X_1. \tag{3.2}$$

A global left-invariant frame can be expressed by vector fields $e_i : g \mapsto \mathrm{d}\,L_g|_e\,(X_i)$, where L_g denotes the left multiplication by $g \in \mathrm{SU}(2)$ and $e \in \mathrm{SU}(2)$ is the identity. Let $a : \mathbb{R} \to \mathbb{R}$ be a smooth positive function. We introduce the Berger metric g_t depending on $t \in \mathbb{R}$ as follows:

$$g_t := a^2(t)\, e_1^* \otimes e_1^* + e_2^* \otimes e_2^* + e_3^* \otimes e_3^*.$$

It is easy to see that g_t is left-invariant by definition. Notice that $a(t) = 1$ gives the standard metric on S^3 with sectional curvature $K \equiv 1$.

Now, we put $X := \mathbb{R} \times S^3$ with metric $g := -\mathrm{d}t^2 + g_t$. For any $t \in \mathbb{R}$ we define the compact Cauchy hypersurface $\Sigma_t := \{t\} \times S^3$ equipped with the Berger metric g_t. As before, let $t_- < t_+$ and $\Sigma_\pm := \Sigma_{t_\pm}$, and we write $M = [t_-, t_+] \times S^3$. Since we will be dealing with a tedious calculation, we consider E to be the trivial complex line bundle with trivial connection ∇^E so that there is no external field. Moreover, we assume $a \equiv$ const. in a neighborhood of $t_\pm$ respectively, in order to guarantee a product structure near $\Sigma_- \sqcup \Sigma_+$.

The Integral Term. It is readily checked that $(\frac{\partial}{\partial t}, e_1, e_2, e_3)$ is a global oriented frame on X. We will denote the corresponding dual frame by $(\mathrm{d}t, \theta^1, \theta^2, \theta^3)$. Then obviously, $(\frac{\partial}{\partial t}, \frac{e_1}{a}, e_2, e_3)$ is an orthonormal oriented frame relative to g and again, we make use of the Koszul formula in order to get the connection 1-form ω on X, compare eq. (2.3). On recalling the commutator relations in eq. (3.2), we observe that

$$\left[\frac{\partial}{\partial t}, e_i\right] = 0,$$
$$\left[\frac{e_1}{a}, \frac{\partial}{\partial t}\right] = \dot{a}\frac{e_1}{a^2}$$

for $i = 1, 2, 3$. With this, we obtain

$$\omega = \begin{pmatrix} 0 & -\dot{a}\,\theta^1 & 0 & 0 \\ \dot{a}\,\theta^1 & 0 & -a\,\theta^3 & a\,\theta^2 \\ 0 & a\,\theta^3 & 0 & -\left(2-a^2\right)\,\theta^1 \\ 0 & -a\,\theta^2 & \left(2-a^2\right)\,\theta^1 & 0 \end{pmatrix}. \tag{3.3}$$

Note that the exterior derivative of each θ^i is given by $\mathrm{d}\theta^i = -2\varepsilon_{ijk}\,\theta^j \wedge \theta^k$, where ε_{ijk} denotes the Levi-Civita symbol, $i = 1, 2, 3$. On using $\Omega = \mathrm{d}\omega + \omega \wedge \omega$ again, we calculate the curvature 2-form to be

$$\Omega = \begin{pmatrix} 0 & \substack{-\ddot{a}\,\mathrm{d}t\wedge\theta^1 \\ +2\dot{a}\,\theta^2\wedge\theta^3} & a\dot{a}\,\theta^1 \wedge \theta^3 & -a\dot{a}\,\theta^1 \wedge \theta^2 \\ \substack{\ddot{a}\,\mathrm{d}t\wedge\theta^1 \\ -2\dot{a}\,\theta^2\wedge\theta^3} & 0 & \substack{-\dot{a}\,\mathrm{d}t\wedge\theta^3 \\ +a^3\,\theta^1\wedge\theta^2} & \substack{\dot{a}\,\mathrm{d}t\wedge\theta^2 \\ +a^3\,\theta^1\wedge\theta^3} \\ -a\dot{a}\,\theta^1 \wedge \theta^3 & \substack{\dot{a}\,\mathrm{d}t\wedge\theta^3 \\ -a^3\,\theta^1\wedge\theta^2} & 0 & \substack{2a\dot{a}\,\mathrm{d}t\wedge\theta^1 \\ +(4-3a^2)\,\theta^2\wedge\theta^3} \\ a\dot{a}\,\theta^1 \wedge \theta^2 & \substack{-\dot{a}\,\mathrm{d}t\wedge\theta^2 \\ -a^3\,\theta^1\wedge\theta^3} & \substack{-2a\dot{a}\,\mathrm{d}t\wedge\theta^1 \\ -(4-3a^2)\,\theta^2\wedge\theta^3} & 0 \end{pmatrix}.$$

A lenghty calculation using Laplace's expansion and Sarrus' rule applied to eq. (2.5) leads to

$$p\left(\nabla^{TX}\right) = 1 - \frac{1}{\pi^2}\left[\ddot{a} + 4a(a^2-1)\right]\frac{\dot{a}}{a}\,\mathrm{dvol}_X,$$

where naturally $\mathrm{dvol}_X = a\,\mathrm{d}t\wedge\theta^1\wedge\theta^2\wedge\theta^3$. Hence we have $\widehat{A}(\nabla^{TX}) = 1 + \frac{1}{24\pi^2}\left[\ddot{a} + 4a(a^2-1)\right]\frac{\dot{a}}{a}$ dvol_X so that

$$\begin{aligned} \int_M \widehat{A}\left(\nabla^{TX}\right) &= \frac{1}{24\pi^2}\int_{t_-}^{t_+}\left[\ddot{a} + 4a^3(t) - 4a(t)\right]\dot{a}(t)\,\mathrm{d}t\int_{S^3}\mathrm{dvol}_{S^3} \\ &= \frac{1}{12}\left[\frac{\dot{a}^2(t_+)}{2} - \frac{\dot{a}^2(t_-)}{2} + a^4(t_+) - 2a^2(t_+) - a^4(t_-) + 2a^2(t_-)\right], \end{aligned}$$

and since a is required constant near $t_\pm$, we finally obtain

$$\int_M \widehat{A}\left(\nabla^{TX}\right) = \frac{1}{12}\left[a^4(t_+) - 2a^2(t_+) - a^4(t_-) + 2a^2(t_-)\right]. \tag{3.4}$$

The Dirac Spectrum. Let us first investigate the inherent spin structures on X. We have already seen that $S^3 \cong \mathrm{SU}(2)$ is parallelizable. Furthermore, it is well-known that S^3 is simply connected. Altogether, S^3, and thus $X = \mathbb{R} \times S^3$, admits a unique spin structure, namely the trivial one. So we have $\mathrm{Spin}(S^3) = S^3 \times \mathrm{Spin}(3)$, where $\mathrm{Spin}(3) \cong \mathrm{SU}(2) \cong S^3$. Smooth sections of the spinor bundle $S\Sigma_t$ are therefore given by smooth $\mathbb{C}^2$-valued functions on S^3.

Now, we fix an oriented orthonormal global frame with respect to g_t, particularly

$$E_1(t) := \frac{1}{a(t)} e_1, \quad E_2(t) := e_2, \quad E_3(t) := e_3.$$

Furthermore, we choose the spinor representation in such a way that each $E_k(t)$ acts via Clifford multiplication as X_k, cf. eqs. (3.1). In comparison to eq. (3.3), the connection 1-form ω_{Σ_t} on Σ_t relative to that basis has the form

$$\omega_{\Sigma_t} = \begin{pmatrix} 0 & -a(t)\,\theta^3 & a(t)\,\theta^2 \\ a(t)\,\theta^3 & 0 & -\left(2 - a^2(t)\right)\,\theta^1 \\ -a(t)\,\theta^2 & \left(2 - a^2(t)\right)\,\theta^1 & 0 \end{pmatrix}.$$

Due to the previous considerations, we put $\psi_1 := \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$ and $\psi_2 := \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)$ as the standard orthonormal spinor basis and get

$$\nabla^{S\Sigma_t}\psi_\alpha = \frac{1}{4}\sum_{k,l=1}^{3} \left(\omega_{\Sigma_t}\right)_k^l\, X_k \cdot X_l \cdot \psi_\alpha$$

which directly leads to the Dirac operator $\not{\nabla}_{\Sigma_t}$ acting on ψ_α as

$$\begin{aligned} \not{\nabla}_{\Sigma_t}\psi_\alpha &= \sum_{k=1}^{3} X_k \cdot \nabla^{S\Sigma_t}_{E_k(t)}\psi_\alpha \\ &= \frac{2 + a^2(t)}{2a(t)}\, X_1 X_2 X_3\, \psi_\alpha \\ &= -\frac{2 + a^2(t)}{2a(t)}\, \psi_\alpha, \end{aligned} \tag{3.5}$$

where $\alpha = 1, 2$. More generally, put $\psi := c_1\psi_1 + c_2\psi_2$, where $c_\alpha : S^3 \to \mathbb{C}$ is a smooth function for $\alpha = 1, 2$. Then, using the Leibniz rule and eq. (3.5), we get

$$\begin{aligned}\not{\nabla}_{\Sigma_t}\psi &= \sum_{k=1}^{3} X_k \cdot \nabla^{S\Sigma_t}_{E_k(t)}\psi \\ &= \sum_{\alpha=1}^{2}\left(\sum_{k=1}^{3} \partial_{E_k(t)}c_\alpha\, X_k - c_\alpha \frac{2+a^2(t)}{2a(t)}\right)\psi_\alpha\end{aligned}$$

which is, in short,

$$\not{\nabla}_{\Sigma_t} = \begin{pmatrix} \frac{\mathrm{i}}{a(t)}\partial_{e_1} & \mathrm{i}\partial_{e_3} + \partial_{e_2} \\ \mathrm{i}\partial_{e_3} - \partial_{e_2} & -\frac{\mathrm{i}}{a(t)}\partial_{e_1} \end{pmatrix} - \frac{2+a^2(t)}{2a(t)}. \tag{3.6}$$

It is convenient to decompose $L^2(S^3, \mathbb{C}^2)$ into suitable $\not{\nabla}_{\Sigma_t}$-invariant subspaces. Since S^3 is a compact Lie group, the Peter-Weyl decomposition

$$L^2\left(\mathrm{SU}(2), \mathbb{C}\right) = \overline{\bigoplus_{\pi\in\widehat{\mathrm{SU}}(2)} \mathcal{E}_\pi}$$

is a useful approach, cf. Theorem 5.12 in [Fol16, p. 143]. Here $\widehat{\mathrm{SU}}(2)$ denotes the set of irreducible unitary representations up to equivalence. Consider $\pi \in \widehat{\mathrm{SU}}(2)$ and V_π its finite dimensional representation space. Then, $\mathcal{E}_\pi$ is the vector space spanned by orthonormal functions

$$g \mapsto h_{e_i^\pi, e_j^\pi}(g) := \left\langle \pi(g)e_i^\pi, e_j^\pi \right\rangle_{V_\pi},$$

called *matrix coefficients*, where $\{e_i^\pi\}_{1\le i\le \dim(V_\pi)}$ is an orthonormal basis of V_π relative to its inner product $\langle\cdot,\cdot\rangle_{V_\pi}$ and $1 \le i, j \le \dim(V_\pi)$. In order to apply the Peter-Weyl decomposition to the Dirac operator, we need to know how left-invariant vector fields act on the given subspaces. The following lemma reduces this problem to linear algebra.

Lemma 3.1. *Let X be a left-invariant vector field on a compact Lie group G. Moreover, let $\pi : G \to Aut(V_\pi)$ be a smooth unitary irreducible representation and V_π its finite dimensional representation space. Then, the subspace $\mathcal{E}_\pi$ can be identified with $V_\pi \otimes V_\pi$ and X acts on $V_\pi \otimes V_\pi$ as*

$$v \otimes w \mapsto \mathrm{d}\pi(X_e)v \otimes w,$$

where $\mathrm{d}\pi(X_e) \in End(V_\pi)$ and $e \in G$ denotes the identity.

Proof. The identification $\mathcal{E}_\pi \cong V_\pi \otimes V_\pi$ is given by the isomorphism $v \otimes w \mapsto h_{v,w}$. Now, assume $v \otimes w \in V_\pi \otimes V_\pi$. Then we have for any $g \in G$

$$\begin{aligned}\partial_X h_{v,w}(g) &= \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} \langle \pi\left(g \cdot \exp(X_e t)\right) v, w\rangle_{V_\pi} \\ &= \left\langle \pi(g) \, \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} \pi\left(\exp(X_e t)\right) v, w\right\rangle_{V_\pi} \\ &= \langle \pi(g)\mathrm{d}\pi(X_e)v, w\rangle_{V_\pi} \\ &= h_{\mathrm{d}\pi(X_e)v,w}(g)\end{aligned}$$

from which follows the claim. □

The irreducible representations of SU(2) are given by homogeneous polynomials in two complex variables of degree m where $m \geq 0$, cf. [Fol16, pp. 149–156]. We will denote them by V_m. Indeed, we have $\dim(V_m) = m + 1$ for each $m \geq 0$ and the corresponding unitary irreducible representations $\pi_m : \mathrm{SU}(2) \to \mathrm{Aut}(V_m)$ are naturally given by

$$\left(\pi_m(g)p\right)(z) = p\left(z \cdot g\right), \quad z \in \mathbb{C}^2$$

for any polynomial $p \in V_m$ and any group element $g \in \mathrm{SU}(2)$.

A remarkable fact that is of significance to us is that the subspaces $\mathcal{E}_m$ generated by the matrix coefficients consist of spherical harmonics. That is, each $\mathcal{E}_m$ is an eigenspace of the Laplacian $\Delta = \partial_{e_1}^2 + \partial_{e_2}^2 + \partial_{e_3}^2$, cf. [Fol16, p. 155]. A simple calculation using eqs. (3.2) shows that $\not{\nabla}_{\Sigma_t}$ of eq. (3.6) and $\left(\begin{smallmatrix}\Delta & 0\\ 0 & \Delta\end{smallmatrix}\right)$ commute, so $\not{\nabla}_{\Sigma_t}$ leaves the subspaces $\mathcal{E}_m \times \mathcal{E}_m$ invariant. Clearly, π_m is a smooth map for each $m \geq 0$ and according to Lemma 3.1, it is enough to investigate the actions of Lie algebra elements as endomorphisms on V_m to deal with the Dirac operator. Referring to [Hal15, p. 83], we have the following operations given on V_m:

$$H := \mathrm{d}\pi_m(\mathrm{i}X_1) = z_1 \frac{\partial}{\partial z_1} - z_2 \frac{\partial}{\partial z_2}, \tag{3.7}$$

$$2L_- := \mathrm{d}\pi_m(\mathrm{i}X_3 - X_2) = 2z_2 \frac{\partial}{\partial z_1}, \tag{3.8}$$

$$2L_+ := \mathrm{d}\pi_m(\mathrm{i}X_3 + X_2) = 2z_1 \frac{\partial}{\partial z_2}. \tag{3.9}$$

Therefore, via eq. (3.6), $\not{\nabla}_{\Sigma_t}$ acts on $\mathcal{E}_m \times \mathcal{E}_m$ as

$$Q := \begin{pmatrix} \frac{H}{a(t)} & 2L_+ \\ 2L_- & -\frac{H}{a(t)} \end{pmatrix} - \frac{2 + a^2(t)}{2a(t)}$$

on $V_m \times V_m$ from Lemma 3.1. The case $m = 0$ is easy to handle since V_0 consists of constant functions and H, L_- plus L_+ are zero; and we obviously obtain the eigenvalue $-\frac{2+a^2(t)}{2a(t)}$ with multiplicity 2. Now, assume $m \geq 1$. We define $p_k(z) := z_1^k z_2^{m-k}$ for any $z \in \mathbb{C}^2$, $k = 0, 1, \ldots, m$. Clearly, $(p_k)_{0 \leq k \leq m}$ is a basis of V_m. Then, equation (3.7) gives

$$H p_k = (2k - m)\, p_k$$

for $k = 0, 1, \ldots, m$ and eqs. (3.8) & (3.9) lead to

$$\begin{aligned} L_- p_k &= k\, p_{k-1}, \\ L_+ p_{k-1} &= (m - k + 1)\, p_k \end{aligned}$$

for $k = 1, 2, \ldots, m$, whereas $L_- p_0 = L_+ p_m = 0$. We find the two-dimensional subspaces

$$\begin{aligned} P_0 &:= \operatorname{span}\left\{ \left(\begin{smallmatrix} p_0 \\ 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 \\ p_m \end{smallmatrix}\right) \right\}, \\ P_k &:= \operatorname{span}\left\{ \left(\begin{smallmatrix} p_k \\ 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 \\ p_{k-1} \end{smallmatrix}\right) \right\}, \end{aligned}$$

$k = 1, \ldots, m$ are invariant under Q and induce the splitting $V_m \times V_m = \bigoplus_{k=0}^m P_k$. Restricted to these subspaces, we obtain

$$Q|_{P_0} = \begin{pmatrix} -\frac{m}{a(t)} & 0 \\ 0 & -\frac{m}{a(t)} \end{pmatrix} - \frac{2 + a^2(t)}{2a(t)},$$

which already has a diagonal form and

$$Q|_{P_k} = \begin{pmatrix} \frac{2k-m}{a(t)} & 2\,(m - k + 1) \\ 2k & -\frac{2k-m-2}{a(t)} \end{pmatrix} - \frac{2 + a^2(t)}{2a(t)}, \tag{3.10}$$

which needs to be diagonalized. Neglecting the second shift term in eq. (3.10), the characteristic polynomial χ of the matrix in the first term is computed as

$$\chi(\lambda) = \lambda^2 - \frac{2}{a(t)}\lambda - \frac{(2k - m)^2 - 2(2k - m)}{a^2(t)} - 4k(m - k + 1)$$

whose zeros are $\lambda = -[a(t)]^{-1} \pm \sqrt{[a(t)]^{-2}(m+1-2k)^2 + 4k(m+1-k)}$. On combining these observations, the eigenvalues of Q on $V_m \times V_m$ for each $m \geq 1$ are therefore given by

$$\begin{aligned} -\frac{m+1}{a(t)} - \frac{a(t)}{2} \quad &\text{with multiplicity} \quad 2, \\ -\frac{a(t)}{2} \pm \frac{1}{a(t)}\sqrt{(m+1-2k)^2 + 4k(m+1-k)\,a^2(t)} \quad &\text{with multiplicity} \quad 1, \end{aligned}$$

where $k = 1, \ldots, m$. Bearing in mind that $\not\nabla_{\Sigma_t}$ actually operates on the spaces $\mathcal{E}_m$ given by the tensor product $V_m \otimes V_m$, we can finally state the result. In order to simplify, we substitute $p := m+1-k$ and $q := k$ in the following.

Theorem 3.2 (Dirac Spectrum of Berger's Sphere)**.** *Assume the conditions listed at the beginning of this section. Then, the eigenvalues of the untwisted Dirac operator $\not\nabla_{\Sigma_t}$ on Σ_t are obtained as follows:*

$$\begin{aligned} -\frac{p}{a(t)} - \frac{a(t)}{2} \quad &\textit{with multiplicity} \quad 2p, \\ -\frac{a(t)}{2} \pm \frac{1}{a(t)}\sqrt{4pq\,a^2(t) + (p-q)^2} \quad &\textit{with multiplicity} \quad p+q, \end{aligned}$$

where $p, q > 0$ are integers.

Eta Invariant and Result. Since we want to consider a non-trivial spectral flow, we take a look at the null space of $\not\nabla_{\Sigma_t}$. By Theorem 3.2, there are zero eigenvalues if and only if

$$a^2(t) = 2\sqrt{4pq\,a^2(t) + (p-q)^2}.$$

So, if we choose $a(t) = 4m$ for a positive integer m, then a solution to this equation is given by $p = q = m$. For simplifications we investigate the case $m = 1$ and put $a(t_-)$ slightly smaller and $a(t_+)$ slightly bigger than 4 such that $h(\not\nabla_{\Sigma_\pm}) = 0$. Then, all eigenvalues take negative signs except for

$$\begin{aligned} 0 < -\frac{a^2(t_-)}{2} + \sqrt{4pq\,a^2(t_-) + (p-q)^2} \quad &\text{for } p, q > 0, \\ 0 < -\frac{a^2(t_+)}{2} + \sqrt{4pq\,a^2(t_+) + (p-q)^2} \quad &\text{for } p, q > 1. \end{aligned}$$

Consequently, the eta functions $\eta_{\Sigma_\pm}$, cf. eq. (1.5), satisfy

$$[a(t_-)]^{-s}\,\eta_{\Sigma_-}(s) = \zeta_1^-(s) + \zeta_2^-(s), \tag{3.11}$$

$$[a(t_+)]^{-s}\,\eta_{\Sigma_+}(s) = \zeta_1^+(s) + \zeta_2^+(s) + \zeta_3^+(s), \tag{3.12}$$

where we have, for $\mathrm{Re}(s)$ being sufficiently large,

$$\begin{aligned}
\zeta_1^\pm(s) &= -\sum_{p>0} 2p\left(p + \frac{a^2(t_\pm)}{2}\right)^{-s}, \\
\zeta_2^-(s) &= -\sum_{p,q>0}(p+q)\left[\left(\frac{a^2(t_-)}{2} + \sqrt{4pq\,a^2(t_-) + (p-q)^2}\right)^{-s}\right. \\
&\qquad\left. - \left(-\frac{a^2(t_-)}{2} + \sqrt{4pq\,a^2(t_-) + (p-q)^2}\right)^{-s}\right], \qquad (3.13) \\
\zeta_2^+(s) &= -\sum_{p,q>1}(p+q)\left[\left(\frac{a^2(t_+)}{2} + \sqrt{4pq\,a^2(t_+) + (p-q)^2}\right)^{-s}\right. \\
&\qquad\left. - \left(-\frac{a^2(t_+)}{2} + \sqrt{4pq\,a^2(t_+) + (p-q)^2}\right)^{-s}\right], \qquad (3.14) \\
\zeta_3^+(s) &= -2\left(\frac{a^2(t_+)}{2} - 2a(t_+)\right)^{-s} - 2\left(\frac{a^2(t_+)}{2} + 2a(t_+)\right)^{-s}.
\end{aligned}$$

The first terms $\zeta_1^\pm$ are easy to handle since

$$\begin{aligned}
\sum_{p>0} p\,(p+x)^{-s} &= \sum_{p>0}(p+x)^{-(s-1)} - x\sum_{p>0}(p+x)^{-s} \\
&= \zeta_H(s-1,x) - x\zeta_H(s,x),
\end{aligned}$$

where $\zeta_H(\cdot,\cdot)$ is the Hurwitz zeta function and $\mathrm{Re}(x) > 0$. After analytic continuation and evaluation at $s = 0$ this becomes

$$\begin{aligned}
\zeta_H(-1,x) - x\zeta_H(0,x) &= -\frac{B_2(x)}{2} + xB_1(x) \\
&= \frac{6x^2-1}{12},
\end{aligned}$$

where $B_n(x)$ denotes the n-th Bernoulli polynomial in x. Hence putting $x = \frac{a^2(t_\pm)}{2}$, we get

$$\zeta_1^\pm(0) = \frac{2 - 3a^4(t_\pm)}{12}. \tag{3.15}$$

The very last term ζ_3^+ is also straightforward and yields

$$\zeta_3^+(0) = -4. \tag{3.16}$$

However, the computation of the second terms $\zeta_2^\pm$ require greater effort. In order to deal with that, we need to state a lemma first.

Lemma 3.3. *Consider the multiple zeta function*

$$Z_r(s) := \sum_{p,q>0} (p+q) \left(4pqr^2 + (p-q)^2\right)^{-s} \tag{3.17}$$

with $r \neq 0$. *Then,* $Z_r(s)$ *is holomorphic on* $\mathrm{Re}(s) > 0$ *except for simple poles at* $s = 1$, $s = \frac{1}{2}$ *and* $s = \frac{3}{2}$ *with residues*

$$\mathrm{Res}\,(Z_r, 1) = -\frac{1}{2}, \quad \mathrm{Res}\left(Z_r, \frac{1}{2}\right) = \frac{r^2-1}{6}, \quad \mathrm{Res}\left(Z_r, \frac{3}{2}\right) = \frac{1}{2r^2}. \tag{3.18}$$

Proof. Fix $r \neq 0$. Then, Z_r can be written in terms of an integral, namely

$$Z_r(s) = \frac{1}{\Gamma(s)} \int_0^\infty u^{s-\frac{3}{2}} \sum_{p,q>0} f\left(\sqrt{u}p, \sqrt{u}q\right) \mathrm{d}u, \tag{3.19}$$

where Γ is the well-known gamma function and $f(x,y) = (x+y)\,\mathrm{e}^{-4xyr^2-(x-y)^2}$. According to [Zag77], we have the following asymptotic expansion from the Euler-Maclaurin formula for $t \to 0^+$

$$\begin{aligned}\sum_{p,q>0} f(tp,tq) \sim \frac{1}{t^2}\int_{\mathbb{R}_+^2} f(x,y)\,\mathrm{d}x\mathrm{d}y + \sum_{m,n\geq 0} \beta_m\beta_n f^{(m,n)}(0,0)t^{m+n} \\ + \frac{1}{t}\sum_{m\geq 0}\beta_m t^m \left[\int_0^\infty f^{(0,m)}(x,0)\,\mathrm{d}x + \int_0^\infty f^{(m,0)}(0,y)\,\mathrm{d}y\right]\end{aligned} \tag{3.20}$$

with $\beta_m = (-1)^m \frac{B_{m+1}}{(m+1)!}$. Here B_m denotes the m-th Bernoulli number. Thus we have

$$\sum_{p,q>0} f(\sqrt{u}p, \sqrt{u}q) = \frac{a}{u} + \frac{b}{\sqrt{u}} + c + \mathcal{O}(\sqrt{u})$$

as u tends to zero from above. The coefficients a, b, c emerge from the expansion (3.20). Putting that into eq. (3.19), we obtain

$$\begin{aligned} Z_r(s) &= \frac{1}{\Gamma(s)} \int_0^1 u^{s-\frac{3}{2}} \left(\frac{a}{u} + \frac{b}{\sqrt{u}} + c \right) \mathrm{d}u + g(s) \\ &= \frac{1}{\Gamma(s)} \left(\frac{a}{s-\frac{3}{2}} + \frac{b}{s-1} + \frac{c}{s-\frac{1}{2}} \right) + g(s), \end{aligned}$$

for $\mathrm{Re}(s) > \frac{3}{2}$. Clearly, g is as an analytic function on $\mathrm{Re}(s) > 0$. A straightforward computation consisting of elementary integrals leads to the coefficients a, b, c and further to the residues. □

Now, we apply a Taylor expansion to $\zeta_2^{\pm}(s)$, cf. equations (3.13) and (3.14), in a domain of $s \in \mathbb{C}$ in which $\zeta_2^{\pm}$ is analytic so that we get

$$\begin{aligned} \zeta_2^{\pm}(s) = 2s \frac{a^2(t_\pm)}{2} F^{\pm}\left(\frac{s+1}{2}\right) \\ + \frac{2s(s+1)(s+2)}{3!} \left(\frac{a^2(t_\pm)}{2}\right)^3 F^{\pm}\left(\frac{s+3}{2}\right) + g^{\pm}(s). \end{aligned}$$

Here, $F^{\pm}$ is obtained from eq. (3.17); in particular we have $F^{-}(s) = Z_{a(t_-)}(s)$ and further by considering the index shift $F^{+}(s) = Z_{a(t_+)}(s) - 2\left[4a^2(t_+)\right]^{-s}$. Since $F^{\pm}$ is analytic for $\mathrm{Re}(s) > \frac{3}{2}$, the remainder $g^{\pm}$ is an analytic function in $\mathrm{Re}(s) > -2$ tending to zero as $s \to 0$. Taking the residues into account, we finally get

$$\begin{aligned} \zeta_2^{\pm}(0) &= 2a^2(t_\pm)\,\mathrm{Res}\left(Z_{a(t_\pm)}, \frac{1}{2}\right) + \frac{a^6(t_\pm)}{6}\,\mathrm{Res}\left(Z_{a(t_\pm)}, \frac{3}{2}\right) \\ &\overset{(3.18)}{=} \frac{5a^4(t_\pm) - 4a^2(t_\pm)}{12}. \end{aligned} \tag{3.21}$$

Now we combine all our calculations together. Remember the formulas (3.11) and (3.12) for $\eta_{\Sigma_\pm}$. Evaluated at $s = 0$, their particular summands are given by eqs. (3.15), (3.16) and (3.21) so that

$$\eta(\nabla\!\!\!\!/_{\Sigma_-}) = \frac{a^4(t_-) - 2a^2(t_-) + 1}{6},$$
$$\eta(\nabla\!\!\!\!/_{\Sigma_+}) = \frac{a^4(t_+) - 2a^2(t_+) + 1}{6} - 4.$$

Notice that we have chosen $a(t_\pm)$ in such a way that $h(\nabla\!\!\!\!/_{\Sigma_\pm}) = 0$ and remember the $\widehat{A}$-genus in eq. (3.4). Altogether, the relative chiral charge, cf. eq. (1.4), takes the value

$$Q_{\mathrm{chir}}^{\omega_{\Sigma_-},\omega_{\Sigma_+}} = 4. \tag{3.22}$$

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